

Recall from last time:

Theorem (Landau)

Let $f \in \mathcal{H}$ with $\sigma_c(f) < \infty$ and $f(n) \geq 0$,
 $\forall n \in \mathbb{N}$. Then $L_f(s)$ has singularity (i.e. not
holomorphic) at $s = \sigma_c(f)$.

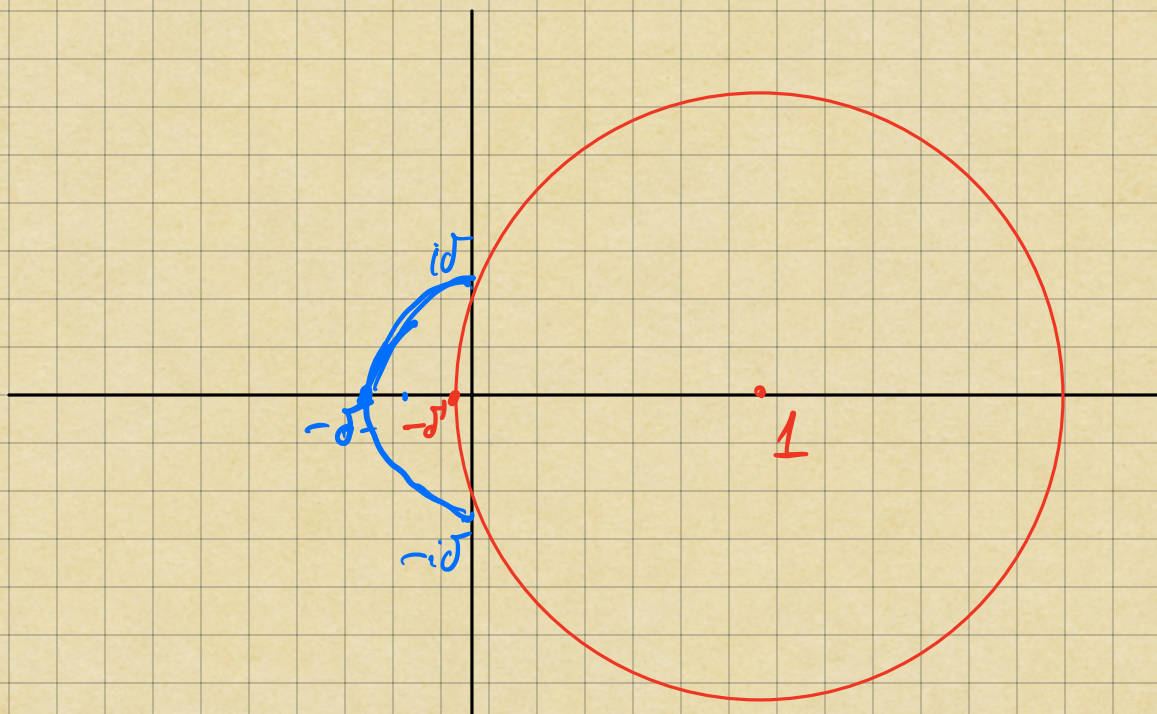
Proof:

By replacing $f(n)$ with $f(n)n^{-\sigma_c(f)}$, we may
assume $\sigma_c(f) = 0$.

Suppose for contradiction $L_f(s)$ is analytic at $s=0$.
Then it is holomorphic in a neighbourhood of $s=0$.

We may assume $L_f(s)$ analytic in

$$D := \{s: \operatorname{Re}(s) > 0\} \cup \{s: |s| < \delta\}.$$



We write the Taylor expansion of $L_f(s)$ at $s=1$:

$$L_f(s) = \sum_{k=0}^{\infty} \frac{L_f^{(k)}(1)}{k!} (s-1)^k \quad (\otimes)$$

One can check: $L_f^{(k)}(1) = \sum_{n=1}^{\infty} \frac{f(n)}{n} (-\log n)^k$.

Also, the radius of convergence of the power series (\otimes) is the distance from 1 to nearest singularity of $L_f(s)$.

Since $L_f(s)$ analytic in D , nearest points to 1 not in D are $\pm i\delta$, hence radius of convergence $\geq \sqrt{1+\delta^2} = 1+\delta'$.

Let $-\delta' < -\sigma < 0$. Then

$$L_f(-\sigma) = \sum_{k=0}^{\infty} \frac{(-\sigma-1)^k}{k!} \sum_{n=1}^{\infty} \frac{(-\log n)^k \cdot f(n)}{n}$$

$$= \sum_{k=0}^{\infty} \frac{(1+\sigma)^k}{k!} \sum_{n=1}^{\infty} \frac{(\log n)^k \cdot f(n)}{n}$$

(note this is convergent series of positive numbers, so it can be rearranged)

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} \sum_{k=0}^{\infty} \frac{(1+\sigma)^k (\log n)^k}{k!}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} \exp((1+\sigma) \log n) = \sum_{n=1}^{\infty} f(n) n^{\sigma}$$

But this is divergent, contradiction. \square

Strengthenings of Dirichlet theorem

Theorem let χ be a non-principal character mod q .

Then:

$$(i) \sum_{n \leq x} \frac{\chi(n) \log n}{n} = -L'(1, \chi) + O_{\chi}\left(\frac{\log x}{x}\right).$$

$$(ii) \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} \ll_{\chi} 1$$

$$(iii) \sum_{p \leq x} \frac{\chi(p) \log p}{p} \ll_{\chi} 1.$$

$$(iv) \sum_{p \leq x} \frac{\chi(p)}{p} = \log L(1, \chi) - \sum_p \sum_{l \geq 2} \frac{\chi(p^l)}{p^l} + O_{\chi}\left(\frac{1}{\log x}\right)$$

$$(v) \prod_{p \leq x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = L(1, \chi) + O_{\chi}\left(\frac{1}{\log x}\right).$$

Proof (i) Note that since $L(s, \chi)$ holomorphic for $\operatorname{Re}(s) > 0$,

$$L'(s, \chi) = - \sum_n \frac{\chi(n) \log n}{n^s} \text{ for } \operatorname{Re}(s) > 0.$$

(can take derivatives for holomorphic functions).

$$\text{Hence } \sum_{n \leq x} \frac{\chi(n) \log n}{n} = -L'(1, \chi) + \underbrace{\sum_{n > x} \frac{\chi(n) \log n}{n}}_{E(x)}$$

$$\text{Define } S(x) = \sum_{n \leq x} \chi(n)$$

$$\text{Then } |S(x)| \leq 2 \ll x^{\frac{1}{2}}$$

$$\begin{aligned} \text{Hence } E(x) &= \lim_{T \rightarrow \infty} \sum_{x < n \leq T} \frac{\chi(n) \log n}{n} \\ &= \lim_{T \rightarrow \infty} \left(\underbrace{\frac{S(T) \log T}{T}}_{\rightarrow 0} - \frac{S(x) \log x}{x} + \int_x^T \underbrace{S(t) \left(\frac{\log t}{t}\right)'} dt \right) \end{aligned}$$

$$\text{Hence } E(x) \ll x \frac{\log x}{x} \checkmark \ll \int_x^T \left| \left(\frac{\log t}{t}\right)' \right| dt$$

$$(ii) \text{ Since } \log = \Lambda * \varepsilon, \text{ we have } \xrightarrow{T \rightarrow \infty} \frac{\log x}{x}$$

$$\frac{\chi(n) \log n}{n} = \sum_{d|n} \underbrace{\Lambda(d) \chi(d)}_{g(d)} \cdot \underbrace{\frac{\chi(n/d)}{n/d}}_{h(n/d)} = g * h(n)$$

By convolution method,

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} \sum_{m \leq \frac{x}{d}} \frac{\chi(m)}{m}$$

$$\text{But } \sum_{m \leq y} \frac{\chi(m)}{m} = L(1, \chi) - \sum_{m > y} \frac{\chi(m)}{m}$$

$$\sum_{m \leq y} \frac{\chi(m)}{m} = \lim_{T \rightarrow \infty} \sum_{y < m \leq T} \frac{\chi(m)}{m}$$

$$= \lim_{T \rightarrow \infty} \left(\frac{S(T)}{T} - \frac{S(y)}{y} + \int_y^T \underbrace{\frac{S(t)}{t^2}}_{\ll \frac{1}{xy}} dt \right) \ll \frac{1}{y}.$$

Hence $\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{d \leq x} \frac{\chi(d) \chi(d)}{d} \left(L(1, \chi) + O\left(\frac{d}{x}\right) \right)$

$$= L(1, \chi) \sum_{d \leq x} \frac{\chi(d) \chi(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} \chi(d) \chi(d)\right).$$

$$\left| \sum_{d \leq x} \chi(d) \chi(d) \right| \leq \sum_{d \leq x} \chi(d) = \psi(x) \ll x$$

(by Chebyshev)

$$\Rightarrow -L'(1, \chi) + O_x\left(\frac{\log x}{x}\right) = L(1, \chi) \sum_{d \leq x} \frac{\chi(d) \chi(d)}{d} + O_x(1).$$

Conclusion follows since $L(1, \chi) \neq 0$.

(iii), (iv), (v) exercise (Problem sheet 7)

Corollary: If $(a, q) = 1$, $x \geq 2$

$$(a) \sum_{\substack{n \leq x \\ n \equiv a(q)}} \frac{\chi(n)}{n} = \frac{1}{\phi(q)} \log x + O_q(1);$$

$$(ii) \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_{\phi}(1);$$

(iii) There exists a constant $b(q, a)$ s.t.

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \frac{1}{p} = \frac{1}{\phi(q)} \log \log x + b(q, a) + O_{\phi}\left(\frac{1}{\log x}\right)$$

Think about these as Mertens theorems in arithmetic progressions.

Perron formula

→ one of our most important tools to relate arithmetic properties of $f \in \mathcal{R}$ to $L_f(s)$.

Theorem (Perron)

Let $f \in \mathcal{R}$, $c > \sigma_a(f)$, $c > 0$ and $x > 0$. Then

$$\sum'_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{(c)} L_f(s) x^s \frac{ds}{s},$$

$(c) = \{s \in \mathbb{C} : \operatorname{Re}(s) = c\}$.

where $\sum'_{n \leq x} f(n) = \begin{cases} \sum_{n \leq x} f(n), & \text{if } x \notin \mathbb{Z} \\ \sum_{n < x} f(n) + \frac{f(x)}{2}, & \text{otherwise} \end{cases}$

Remark: Here we define

$$\int_{(c)} L_f(s) x^s \frac{ds}{s} := \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} L_f(s) x^s \frac{ds}{s}.$$

! Not true in general integral converges absolutely, need to be careful!.

At the heart of the theorem is the following lemma:

Central lemma: Let $\delta(y) = \begin{cases} 1, & \text{if } y > 1 \\ 1/2, & \text{if } y = 1 \\ 0, & \text{if } 0 < y < 1. \end{cases}$

For all $c > 0$ and $T \geq 2$, we have

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} ds = \delta(y) + O\left(\frac{y^c}{T|\log y|}\right) \quad y \neq 1.$$

and for $y = 1$: $\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds = \frac{1}{2} + O\left(\frac{c}{T}\right).$

In particular, $\frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} = \delta(y).$

Heuristic proof of Perron:

$$\frac{1}{2\pi i} \int_{(c)} L_f(s) X^s \frac{ds}{s} = \frac{1}{2\pi i} \int_{(c)} \sum_{n=1}^{\infty} f(n) \left(\frac{X}{n}\right)^s \frac{ds}{s} =$$

$$= \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i} \int_{(c)} \left(\frac{X}{n}\right)^s \frac{ds}{s} \quad (\text{using Fubini})$$

$$= \sum_{n=1}^{\infty} f(n) \delta\left(\frac{X}{n}\right) = \sum_{n \leq X} f(n).$$

Proof of central lemma: Define $I_c(y, T) := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s}$.

If $y=1$: $I_c(1, T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds =$
 $= \frac{1}{2\pi} \int_{-T}^T \frac{1}{c+it} dt = \frac{1}{2\pi} \int_{-T}^T \frac{e^{-it}}{c^2+t^2} dt$

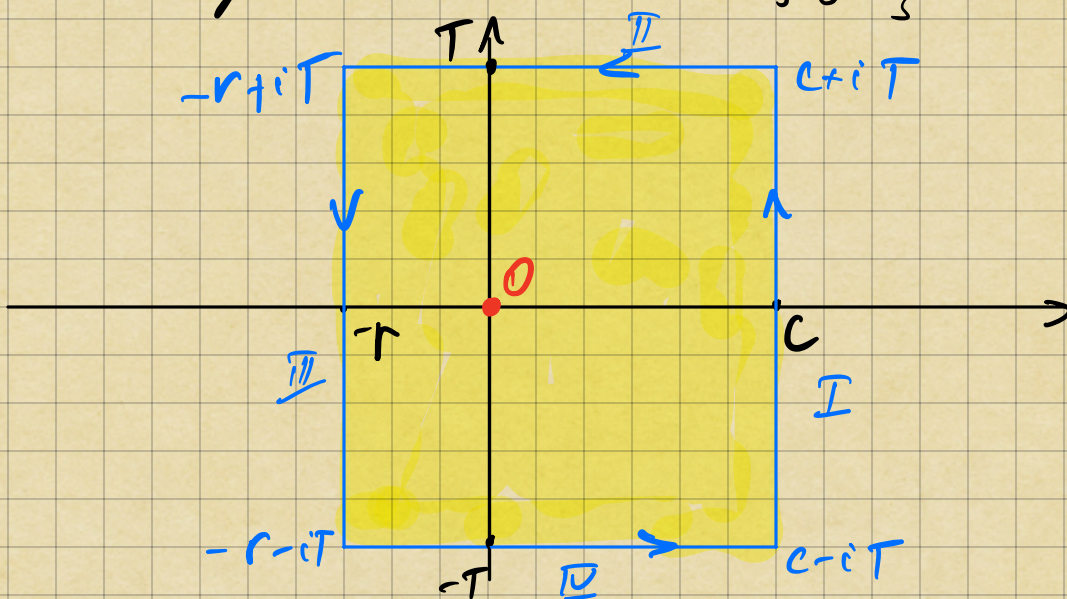
(real part symmetric
 imaginary part cancels) $= \frac{c}{\pi} \int_0^T \frac{1}{c^2+t^2} dt$

$= \frac{1}{\pi} \int_0^{T/c} \frac{dt}{1+t^2} = \frac{1}{\pi} \int_0^{\infty} \frac{dt}{1+t^2} - \frac{1}{\pi} \int_{T/c}^{\infty} \frac{dt}{1+t^2}$
 $= \frac{1}{2} - o\left(\frac{c}{T}\right)$

Case $y \neq 1$: Application of Cauchy's residue theorem.

Note that $\frac{y^s}{s}$ meromorphic in $s \in \mathbb{C}$, with

simple pole at $s=0$ and $\text{Res}_{s=0} \frac{y^s}{s} = 1$.



We integrate along the box with vertices $c-iT$, $c+iT$, $-r+iT$, $-r-iT$, for some $r > 0$.

$$\underbrace{\int_{c-iT}^{c+iT} y^s \frac{ds}{s}}_I + \underbrace{\int_{c+iT}^{-r+iT} y^s \frac{ds}{s}}_{II} + \underbrace{\int_{-r+iT}^{-r-iT} y^s \frac{ds}{s}}_{III} + \underbrace{\int_{-r-iT}^{c-iT} y^s \frac{ds}{s}}_{IV} = 2\pi i \operatorname{Res}_{s=0} \frac{y^s}{s} = 2\pi i.$$

II and IV: we have $|s| \geq T$, $|y^s| \leq y^{\operatorname{Re}(s)}$, so

$$\left| \int_{c-iT}^{-r+iT} y^s \frac{ds}{s} \right| \leq \int_{-r}^c \frac{y^\sigma}{T} d\sigma \leq \frac{1}{T} \int_{-\infty}^c y^\sigma d\sigma = \frac{y^c}{T |\log y|}.$$

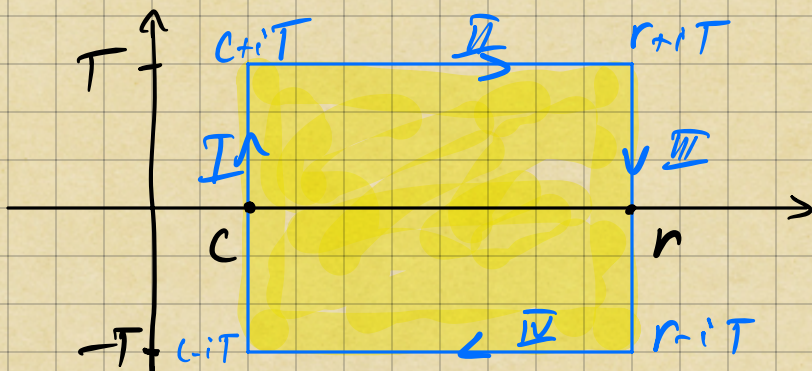
(Similarly for IV).

For III: $|y^s| \leq y^{-r}$, $|s| \geq r$, hence

$$\left| \int_{-r+iT}^{-r-iT} y^s \frac{ds}{s} \right| \leq \int_{-T}^T \frac{y^{-r}}{r} dt \leq \frac{2y^{-r}}{r} \xrightarrow{r \rightarrow \infty} 0.$$

We let $r \rightarrow \infty$, this gives the result.

If $0 < y < 1$: We apply same argument for box $c-iT$, $c+iT$, $r+iT$, $r-iT$, where $r > 1$.



$\frac{y^s}{s}$ is holomorphic in this region, horizontal integrals
have same bound as previous case, and

$$\left| \int_{r+iT}^{r-iT} y^s \frac{ds}{s} \right| \leq \int_{-T}^T \frac{y^r}{r} dt = \frac{2y^r}{r} \rightarrow 0 \text{ as } r \rightarrow \infty. \quad \square$$